PRIME ASSOCIATOR-DEPENDENT RINGS WITH IDEMPOTENT(1)

BY

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1. **Introduction.** Associator-dependent rings as a class were defined by Kleinfeld [2]. They are rings R which satisfy the following identities.

$$\alpha_1(x, y, z)+\alpha_2(y, z, x)+\alpha_3(z, x, y)+\alpha_4(x, z, y)+\alpha_5(z, y, x)+\alpha_6(y, x, z)=0 \tag{1}$$

for fixed α_i in some field of scalars and x, y, z in R where the associator is defined as $(x, y, z)=(xy)z-x(yz)$.

$$(x, x, x) = 0. \tag{2}$$

A linearization of (2) yields (1) with $\alpha_i=1$ for $i=1-6$. Since (2) alone is not strong enough to yield a structure theory for R we assume that (2) does not imply (1).

This class of rings includes, among others, the right and left alternative rings, Lie-admissible rings, rings of type (γ, δ), flexible rings, and antiflexible rings.

Kleinfeld, Kosier, Osborn and Rodabaugh [3] showed that an associator-dependent ring R must satisfy one of the following three identities.

$$\alpha(y, x, x)-(\alpha+1)(x, y, x)+(x, x, y) = 0, \tag{3}$$

$$(y, x, x) = (x, y, x), \tag{4}$$

$$(x, y, z)+(y, z, x)+(z, x, y) = 0. \tag{5}$$

We note that any ring satisfying (4) is anti-isomorphic to one satisfying (3) where $\alpha=0$. It is our purpose in this paper to show that if R is a prime associator-dependent ring satisfying (3) and having an idempotent $e\neq 0, 1$ then when the characteristic of R is prime to 6, R is alternative unless $\alpha=-1, 1, -\frac{1}{2}, -2$. It then follows immediately, from a result of Slater's [6], that R can be embedded in a Cayley-Dickson algebra.

An example to show that this result is false when $\alpha=-1$ can be found in [4]. The residual cases where $\alpha=1, -\frac{1}{2}, -2$ correspond, respectively, to flexible, left alternative, and right alternative rings. With the assumption of Lie-admissibility we have shown in a paper to be published elsewhere that if R is a prime right or left alternative ring with an idempotent $e\neq 0$, 1 and characteristic prime to 6 then

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R is associative and a prime, flexible, power-associative ring R with an idempotent $e \neq 0, 1$ and a Peirce decomposition relative to e with the same restriction on characteristic is associative.

It is worth noting that an arbitrary primitive ring is prime [5]. Hence, by defining a suitable radical, the results of this paper could be extended to semisimple rings.

2. **Preliminary results. Construction of ideals.** Let R be an associator-dependent ring satisfying (3) where $\alpha \neq -1, 1, -\frac{1}{2}, -2$. Furthermore, assume that R has an idempotent $e \neq 0, 1$. If the characteristic of R is prime to 6 the following results hold in R [3].

1. R has the Peirce decomposition $R = R_{11} + R_{10} + R_{01} + R_{00}$ relative to e where x_{ij} belongs to R_{ij} if and only if $ex = ix$ and $xe = jx$ where $i, j = 0, 1$. Moreover, the sum of the submodules R_{ij} is direct.

2. The following relations hold for products of elements from R when $\alpha \neq 0$.

(6) $R_{ij}R_{kp} \subseteq \delta_{jk}R_{ip}$ where $i, j, k, p = 0, 1$ and δ_{jk} is the Kronecker delta, except $R_{ij}^2 \subseteq R_{ji},\ x_{ij}^2 = 0$ where $i = 0, 1$ and $j = 1-i$.

When $\alpha = 0$ all of the above relations hold except

(7) $$R_{ii}R_{ij} \subseteq R_{ij} + R_{jj},\quad R_{ij}R_{ii} \subseteq R_{jj} \text{ and } (x_{ii}y_{ij} - y_{ij}x_{ii}) \text{ is in } R_{ij}.$$

3. When $\alpha = 0$, $K = R_{10}R_{11} + R_{01}R_{00}$ is a trivial ideal of R.

4. The following relations exist for associators of R when $\alpha \neq 0$. When $\alpha = 0$, the same relations hold so long as R has no trivial ideals. Let σ be an arbitrary permutation of the arguments in an associator. Then when $i \neq j$,

(8) $$\begin{aligned}(\sigma x_{ii}, \sigma y_{ij}, \sigma z_{ji}) &= 0,\\ (\sigma x_{ii}, \sigma y_{ij}, \sigma z_{ii}) &= 0,\\ (\sigma x_{ii}, \sigma y_{ji}, \sigma z_{ii}) &= 0,\\ (\sigma x_{ii}, \sigma y_{ij}, \sigma z_{jj}) &= 0,\\ (\sigma x_{ii}, \sigma y_{ii}, \sigma z_{jj}) &= 0,\end{aligned}$$

$(\sigma x, \sigma y, \sigma z) = \operatorname{sgn} \sigma(x, y, z)$ for x, y, z in R unless each of x, y, z has a nonzero component in the same R_{ii} submodule.

We now prove

LEMMA 1. *Let R satisfy* (2) *and* (3) *where $\alpha = 0$. Then if R is a prime ring elements from R satisfy* (6) *and* (8).

Proof. Since R is prime it can contain no trivial ideals. Hence $K = 0$. But then from (7), $x_{ii}y_{ij}$ belongs to R_{ij} and (6) and (8) follow.

For the remainder of this paper we assume that R is a prime ring. Clearly, from (8), R is alternative if and only if R_{11} and R_{00} are alternative. We will prove that, in fact, these subrings must be associative.

It is immediate from (8) that the following identities hold for all x, y, z in R except when x, y, z all have nonzero components in the same R_{ii} submodule.

$$(9)\qquad \begin{aligned} (xy+yx)z &= x(yz)+y(xz), \\ z(xy+yx) &= (zx)y+(zy)x, \\ (zx)y+(yx)z &= z(xy)+y(xz). \end{aligned}$$

The Teichmüller identity, true for an arbitrary ring R, will be of use to us. We state it below.

$$(10)\quad 0 = F(x, y, z, w) = (xy, z, w)-(x, yz, w)+(x, y, zw)-x(y, z, w)-(x, y, z)w.$$

The next two lemmas have been proven for alternative rings [1]. We will show that they are still true under our more general hypotheses.

LEMMA 2. *If B_i is an ideal of R_{ii}, then $D_i=B_i+R_{ji}B_i+B_iR_{ij}+(R_{ji}B_i)R_{ij}$ is an ideal of R, where $i=0, 1$ and $j=1-i$.*

Proof. First, $R_{ii}D_i\subseteq R_{ii}B_i+R_{ii}(B_iR_{ij})$ from (6). However, $R_{ii}B_i\subseteq B_i$ and, from (8), $R_{ii}(B_iR_{ij})\subseteq(R_{ii}B_i)R_{ij}\subseteq B_iR_{ij}$. Hence, $R_{ii}D_i\subseteq D_i$. Similarly, $D_iR_{ii}\subseteq D_i$.

Next, $R_{ji}D_i\subseteq R_{ji}B_i+R_{ji}(R_{ji}B_i)+R_{ji}(B_iR_{ij})$ from (6). But, $R_{ji}(B_iR_{ij})\subseteq(R_{ji}B_i)R_{ij}$ from (8). Also, from (9), $R_{ji}(R_{ji}B_i)\subseteq(R_{ji}^2)B_i+(B_iR_{ji})R_{ji}+B_i(R_{ji}^2)$. Since $(R_{ji}^2)B_i=(B_iR_{ji})R_{ji}=0$, by (6), we conclude that $R_{ji}(R_{ji}B_i)\subseteq B_i(R_{ji}^2)\subseteq B_iR_{ij}$. Thus, $R_{ji}D_i\subseteq D_i$. A similar argument shows that $D_iR_{ij}\subseteq D_i$.

Now, $R_{ij}D_i\subseteq R_{ij}(R_{ji}B_i)+R_{ij}(B_iR_{ij})+R_{ij}[(R_{ji}B_i)R_{ij}]$ from (6). But, $R_{ij}(R_{ji}B_i)\subseteq(R_{ij}R_{ji})B_i\subseteq B_i$ from (8). Since $R_{ij}(B_iR_{ij})\subseteq(R_{ij}B_i)R_{ij}+(R_{ij}^2)B_i+R_{ij}(R_{ij}B_i)$, from (9), and $(R_{ij}B_i)R_{ij}=R_{ij}(R_{ij}B_i)=0$, from (6), we have $R_{ij}(B_iR_{ij})\subseteq(R_{ij}^2)B_i\subseteq R_{ji}B_i$. Finally, by (9), $R_{ij}[(R_{ji}B_i)R_{ij}]\subseteq[R_{ij}(R_{ji}B_i)]R_{ij}+(R_{ij}^2)(R_{ji}B_i)+R_{ij}[R_{ij}(R_{ji}B_i)]$. However, $R_{ij}[R_{ij}(R_{ji}B_i)]=0$ and $[R_{ij}(R_{ji}B_i)]R_{ij}\subseteq[(R_{ij}R_{ji})B_i]R_{ij}\subseteq B_iR_{ij}$ from (6) and (8). Furthermore, $R_{ij}^2(R_{ji}B_i)\subseteq R_{ji}(R_{ji}B_i)\subseteq D_i$ from (6) and the fact that we have already shown that $R_{ji}D_i\subseteq D_i$. Hence, $R_{ij}D_i\subseteq D_i$. A similar argument shows that $D_iR_{ji}\subseteq D_i$.

Finally, we consider $R_{jj}D_i\subseteq R_{jj}(R_{ji}B_i)+R_{jj}[(R_{ji}B_i)R_{ij}]$ from (6). However, $R_{jj}(R_{ji}B_i)\subseteq(R_{jj}R_{ji})B_i\subseteq D_i$ and $R_{jj}[(R_{ji}B_i)R_{ij}]\subseteq[(R_{jj}R_{ji})B_i]R_{ij}\subseteq D_i$ from (8). Thus, $R_{jj}D_i\subseteq D_i$. Similarly, $D_iR_{jj}\subseteq D_i$.

We conclude that RD_i and D_iR are contained in D_i and that D_i is an ideal of R.

LEMMA 3. *If $R_{10}^2R_{10}=R_{01}^2R_{01}=R_{10}R_{10}^2=R_{01}R_{01}^2=0$, then $H=R_{10}^2+R_{01}^2$ is an ideal of R.*

Proof. First, consider $R_{ij}^2R\subseteq R_{ij}^2R_{ii}+R_{ij}^2R_{ij}+R_{ij}^2R_{ji}+R_{ij}^2R_{jj}$. Now, $R_{ij}^2R_{ij}=0$ by hypothesis. Also, $R_{ij}^2R_{jj}=0$ and $R_{ij}^2R_{ji}\subseteq R_{ji}^2$ from (6). Since

$$R_{ij}^2R_{ii}\subseteq R_{ij}(R_{ij}R_{ii})+R_{ij}(R_{ii}R_{ij})+(R_{ij}R_{ii})R_{ij},$$

from (9), and $R_{ij}(R_{ij}R_{ii})=(R_{ij}R_{ii})R_{ij}=0$ by (8), we conclude that $R_{ij}^2R_{ii}\subseteq R_{ij}^2$. Therefore $HR\subseteq H$. A similar argument yields $RH\subseteq H$. Thus H is an ideal of R.

We now proceed with other lemmas preliminary to our main result.

LEMMA 4. *$L=R_{ij}R_{ji}$ is an ideal in R_{ii} in the nucleus of R_{ii} where $i=0, 1$ and $j=1-i$.*

Proof. It is clear from (6) and (8) that L is an ideal of R_{ii}. We will show that L is in the left nucleus of R and note that a similar argument places L in the middle and right nuclei of R.

Let $x_{ij}\in R_{ij}$, $y_{ji}\in R_{ji}$ and z_{ii}, $w_{ii}\in R_{ii}$. Then

$$(x_{ij}y_{ji}, z_{ii}, w_{ii}) = [(x_{ij}y_{ji})z_{ii}]w_{ii}-(x_{ij}y_{ji})(z_{ii}w_{ii}) = 0$$

by repeated applications of (8). Hence, L is in the left nucleus of R_{ii}.

LEMMA 5. *$R_{ji}^2R_{ji}$ and $R_{ij}R_{ij}^2$ are ideals in the center of R_{ii} when $i=0, 1$ and $j=1-i$.*

Proof. It is immediate from Lemma 4 that $R_{ji}^2R_{ji}$ and $R_{ij}R_{ij}^2$ are in the nucleus of R_{ii}. Also, $(R_{ji}^2R_{ji})R_{ii}\subseteq R_{ji}^2(R_{ji}R_{ii})\subseteq R_{ji}^2R_{ji}$ and $R_{ii}(R_{ij}R_{ij}^2)\subseteq(R_{ii}R_{ij})R_{ij}^2\subseteq R_{ij}R_{ij}^2$ from (6) and (8). Thus, $R_{ji}^2R_{ji}$ is a right ideal and $R_{ij}R_{ij}^2$ a left ideal, respectively, of R_{ii}. Therefore, to prove that they are both ideals in the center of R_{ii}, it suffices to show that they commute with each element in R_{ii}.

Before proceeding, we recall from (6) that $x_{ij}^2=0$ for $i\neq j$. Replacing x_{ij} by $x_{ij}+y_{ij}$ we obtain

$$x_{ij}y_{ij}+y_{ij}x_{ij} = 0. \tag{11}$$

Now, let x_{ji}, y_{ji}, z_{ji} belong to R_{ji} and w_{ii} to R_{ii}. Then

$$[(x_{ji}y_{ji})z_{ji}]w_{ii} = (x_{ji}y_{ji})(z_{ji}w_{ii})$$

from (8). But,

$$(x_{ji}y_{ji})(z_{ji}w_{ii})+[x_{ji}(z_{ji}w_{ii})]y_{ji} = x_{j}\cdot[y_{ji}(z_{ji}w_{ii})]+x_{ji}[(z_{ji}w_{ii})y_{ji}] = 0$$

from (9) and (11). Hence,

$$(x_{ji}y_{ji})(z_{ji}w_{ii}) = -[x_{ji}(z_{ji}w_{ii})]y_{ji} = (x_{ji}, z_{ji}, w_{ii})y_{ji} = -(w_{ii}, z_{ji}, x_{ji})y_{ji}$$

from (6) and (8), and so,

$$[(x_{ji}y_{ji})z_{ji}]w_{ii} = -(w_{ii}, z_{ji}, x_{ji})y_{ji}. \tag{12}$$

However, from (10), we have

$$0 = F(w_{ii}, z_{ji}, x_{ji}, y_{ji}) = (w_{ii}z_{ji}, x_{ji}, y_{ji})-(w_{ii}, z_{ji}x_{ji}, y_{ji})$$
$$+(w_{ii}, z_{ji}, x_{ji}y_{ji})-w_{ii}(z_{ji}, x_{ji}, y_{ji})-(w_{ii}, z_{ji}, x_{ji})y_{ji}.$$

Whence, from (6) and (8), we get

$$w_{ii}(z_{ji}, x_{ji}, y_{ji}) = -(w_{ii}, z_{ji}, x_{ji})y_{ji}. \tag{13}$$

Combining (12) and (13), we obtain,

$$[(x_{ji}y_{ji})z_{ji}]w_{ii} = w_{ii}(z_{ji}, x_{ji}, y_{ji}) = w_{ii}(x_{ji}, y_{ji}, z_{ji}) = w_{ii}[(x_{ji}y_{ji})z_{ji}]$$

from (6) and (8). We conclude that elements from $R_{ji}^2R_{ji}$ commute with elements from R_{ii}. A similar proof shows that elements from $R_{ij}R_{ij}^2$ also commute with elements from R_{ii}.

The next two lemmas deal with arbitrary prime rings.

LEMMA 6. *Let R be a nonassociative prime ring. Then R can contain no nuclear ideals.*

Proof. Let B be an ideal in the nucleus of R. Let b belong to B and x, y, z to R. From (10) we obtain

$$0 = F(b, x, y, z) = (bx, y, z)-(b, xy, z)+(b, x, yz)-b(x, y, z)-(b, x, y)z$$
$$= -b(x, y, z).$$

Further, $b[(x, y, z)w]=[b(x, y, z)]w=0$ where w belongs to R. Hence,

$$B(R, R, R) = B[(R, R, R)R] = 0.$$

However, finite sums of elements of the form (R, R, R) and $(R, R, R)R$ form an ideal in an arbitrary ring. Since R is prime, B must be zero unless R is associative.

LEMMA 7. *Let R be an arbitrary prime ring. Then the set of annihilators of a non-zero element in the center of R is zero.*

Proof. Let $c\neq 0$ belong to the center of R and suppose x belongs to R and $xc=cx=0$. Then $c(xR)=(xc)R=0$ and $c(Rx)=(Rx)c=R(xc)=0$. Therefore the annihilator of c is an ideal of R, call it B. It is clear that B annihilates the ideal generated by c. Since c is assumed nonzero, we conclude that $B=0$.

3. **Main section.** The next lemma is crucial to the proof of our main result.

LEMMA 8. *Let C_i and B_i be ideals in R_{ii} such that $B_iC_i=C_iB_i=0$. Then*

$$D_i = B_i+R_{ji}B_i+B_iR_{ij}+(R_{ji}B_i)R_{ij}$$

and

$$F_i = C_i+R_{ji}C_i+C_iR_{ij}+(R_{ji}C_i)R_{ij}$$

are ideals of R and $D_iF_i=F_iD_i=0$.

Proof. From Lemma 4, F_i and D_i are ideals of R. We have left to show that they annihilate one another. First, from (6) we have $D_iC_i\subseteq B_iC_i+(R_jB_i)C_i$. However, $B_iC_i=0$ by hypothesis and $(R_{ji}B_i)C_i\subseteq R_{ji}(B_iC_i)=0$ from (8). A similar argument applies to C_iD_i, F_iB_i, and B_iF_i and we can conclude that

$$D_iC_i = C_iD_i = F_iB_i = B_iF_i = 0. \tag{14}$$

Next, $D_i(R_{ji}C_i)\subseteq(R_{ji}B_i)(R_{ji}C_i)+(B_iR_{ij})(R_{ji}C_i)+[(R_{ji}B_i)R_{ij}](R_{ji}C_i)$ from (6). From (9), $(R_{ji}B_i)(R_{ji}C_i)\subseteq R_{ji}[B_i(R_{ji}C_i)]+B_i[R_{ji}(R_{ji}C_i)]+(B_iR_{ji})(R_{ji}C_i)$. However, $(B_iR_{ji})(R_{ji}C_i)=R_{ji}[B_i(R_{ji}C_i)]=0$ from (6) and $B_i[R_{ji}(R_{ji}C_i)]=0$ from (14). Hence, $(R_{ji}B_i)(R_{ji}C_i)=0$. Finally, $(B_iR_{ij})(R_{ji}C_i)\subseteq B_i[R_{ij}(R_{ji}C_i)]=0$ and $[(R_{ji}B_i)R_{ij}](R_{ji}C_i)$

$\subseteq\{[(R_{ji}B_i)R_{ij}]R_{ji}\}C_i=0$ from (8) and (14). A similar argument applies to $F_i(R_{ji}B_i)$ and we have,

(15) $$D_i(R_{ji}C_i) = F_i(R_{ji}B_i) = 0.$$

Now, $(R_{ji}C_i)D_i\subseteq(R_{ji}C_i)(B_iR_{ij})$ from (6), (14) and (15). But,

$$(R_{ji}C_i)(B_iR_{ij}) \subseteq R_{ji}[C_i(B_iR_{ij})] = 0$$

from (8) and (15). Hence, as a similar argument applies to $(R_{ji}B_i)F_i$, we obtain

(16) $$(R_{ji}C_i)D_i = (R_{ji}B_i)F_i = 0.$$

Next, $D_i(C_iR_{ij})\subseteq(B_iR_{ij})(C_iR_{ij})$ from (6), (14) and (15). However,

$$(B_iR_{ij})(C_iR_{ij}) \subseteq B_i[R_{ij}(C_iR_{ij})]+R_{ij}[B_i(C_iR_{ij})]+(R_{ij}B_i)(C_iR_{ij})$$

from (9). Since $(R_{ij}B_i)(C_iR_{ij})=0$ from (6) and $B_i[R_{ij}(C_iR_{ij})]=R_{ij}[B_i(C_iR_{ij})]=0$ from (14), we conclude that $(B_iR_{ij})(C_iR_{ij})=0$. A similar argument applies to $F_i(B_iR_{ij})$. Hence

(17) $$D_i(C_iR_{ij}) = F_i(B_iR_{ij}) = 0.$$

We have $(C_iR_{ij})D_i\subseteq(C_iR_{ij})[(R_{ji}B_i)R_{ij}]$ from (14), (15) and (17). However,

$$(C_iR_{ij})[(R_{ji}B_i)R_{ij}] \subseteq C_i\{R_{ij}[(R_{ji}B_i)R_{ij}]\} = 0$$

from (8) and (14). A similar argument applies for $(B_iR_{ij})F_i$ and we obtain

(18) $$(C_iR_{ij})D_i = (B_iR_{ij})F_i = 0.$$

Finally, we consider $[(R_{ji}C_i)R_{ij}]D_i$ and $D_i[(R_{ji}C_i)R_{ij}]$. From (14)–(18) it is clear that we need only consider $[(R_{ji}C_i)R_{ij}][(R_{ji}B_i)R_{ij}]$. But,

$$[(R_{ji}C_i)R_{ij}][(R_{ji}B_i)R_{ij}] \subseteq (R_{ji}C_i)\{R_{ij}[(R_{ji}B_i)R_{ij}]\} = 0$$

from (8) and (17). Hence,

$$[(R_{ji}C_i)R_{ij}]D_i = D_i[(R_{ji}C_i)R_{ij}] = 0.$$

Similarly,

$$[(R_{ji}B_i)R_{ij}]F_i = F_i[(R_{ji}B_i)R_{ij}] = 0.$$

We conclude that $D_iF_i=F_iD_i=0$.

COROLLARY. *If R is a prime ring then the subrings R_{ii}, where $i=0, 1$ are also prime rings.*

Proof. Let C_i and B_i be ideals of R_{ii} such that $C_iB_i=B_iC_i=0$. Then D_i and F_i, constructed as in Lemma 2, are ideals of R such that $D_iF_i=F_iD_i=0$. Since R is prime we conclude that either D_i or F_i is zero. Hence, either B_i or C_i is zero and R_{ii} is a prime ring.

We now proceed to analyze the R_{ii} spaces. Our attention is directed toward the four subspaces $R_{ij}R_{ij}^2$ and $R_{ji}^2R_{ji}$ where $i=0, 1$ and $j=1-i$.

First, suppose $R_{10}^2R_{10}=R_{01}^2R_{01}=R_{10}R_{10}^2=R_{01}R_{01}^2=0$. Then from Lemma 3 $H=R_{10}^2+R_{01}^2$ is an ideal of R.

LEMMA 9. *Let $R_{ij}R_{ij}^2=R_{ji}^2R_{ji}=0$ where $i=0, 1$ and $j=1-i$. Then $H=R_{10}^2+R_{01}^2=0$.*

Proof. We have $R_{ij}^2R_{ji}^2=0$ for $i\neq j$ from (6) and the hypothesis. Also,

$$R_{ij}^2R_{ij}^2 \subseteq R_{ij}(R_{ij}R_{ij}^2)+R_{ij}(R_{ij}^2R_{ij})+(R_{ij}R_{ij}^2)R_{ij}$$

from (9). Since all elements on the right of this inclusion relation vanish, $R_{ij}^2R_{ij}^2=0$. Thus, H is a trivial ideal of R. Since R is a prime ring, it follows that $H=0$.

LEMMA 10. *Let $H=R_{10}^2+R_{01}^2=0$. Then $B=R_{10}R_{01}+R_{10}+R_{01}+R_{01}R_{10}$ is an ideal in the nucleus of R.*

Proof. From (6) and (8) it is evident that B is an ideal of R.

Recall, from (8), that all associators containing at least one argument from R_{ij}, where $i\neq j$, alternate. Thus, to prove that R_{ij} is in the nucleus of R, it suffices to show that R_{ij} is in the left nucleus of R.

To this end, consider associators of the form (R_{ij}, R_{kp}, R_{mn}) where $k, p, m, n=0, 1$. From (8), it is evident that unless two of the arguments are from the R_{ij} or the R_{ji} space the associator vanishes. But, $(R_{ij}, R_{ij}, R_{mn})=(R_{ij}, R_{ji}, R_{ji})=0$ regardless of the value m assumes, from (6) and the fact that $H=0$. Hence, R_{ij} is in the nucleus of R.

From Lemma 4, $R_{ij}R_{ji}$, where $i=0, 1$ and $j=1-i$, is in the nucleus of R_{ii}. Since R_{ij}, R_{ji} are in the nucleus of R and associators of the form $(\sigma R_{ii}, \sigma R_{jj}, \sigma R_{jj})=0$ from (8), where σ is an arbitrary permutation of the three arguments, it follows that $R_{ij}R_{ji}$ is in the nucleus of R.

COROLLARY. *$B=0$ if R is not associative.*

Proof. Immediate from Lemmas 6 and 10.

LEMMA 11. *If $R_{ij}^2R_{ij}=R_{ji}R_{ji}^2=0$ for $i=0, 1$ and $j=1-i$ then R is associative.*

Proof. From Lemma 10, Corollary, we obtain $R=R_{11}+R_{00}$. Since R_{11} and R_{00} are orthogonal subrings of R we conclude that they are, in fact, orthogonal ideals of R. Since $e\neq 0$ belongs to R_{11}, we get $R_{00}=0$. But then $R=R_{11}$ and e becomes the identity for R, contrary to hypothesis. Hence, R must be associative.

It follows that, unless R is associative, at least one of the subspaces $R_{10}^2R_{10}$, $R_{01}^2R_{01}$, $R_{10}R_{10}^2$, $R_{01}R_{01}^2$ must not vanish. We will show that, under this condition, both R_{11} and R_{00} must be associative and R must be alternative.

Assume, without loss of generality, that $R_{01}R_{01}^2\neq 0$. From Lemma 5, $R_{01}R_{01}^2$ is a nonzero ideal in the center of R_{00}. Hence, from Lemma 6, R_{00} must be associative.

Recall, from Lemma 4, that $R_{10}R_{01}$ is an ideal in the nucleus of R_{11}. Unless R_{11} is also associative, $R_{10}R_{01}=0$. Under the assumption that $R_{10}R_{01}=0$, we examine elements of the form $x_{01}y_{10}$ belonging to $R_{01}R_{10}$. From (8), we have

$$(x_{01}y_{10})^2 = [(x_{01}y_{10})x_{01}]y_{10}, \quad \text{and} \quad (x_{01}, y_{10}, x_{01}) = 0.$$

Expanding this associator, we obtain $(x_{01}y_{10})x_{01}-x_{01}(y_{10}x_{01})$. However, $x_{01}(y_{10}x_{01})=0$, and we obtain

$$(x_{01}y_{10})x_{01} = 0 = (x_{01}y_{10})^2.$$

In particular, $[x_{01}(y_{01}z_{01})]^2=0$ for x_{01}, y_{01}, z_{01} in R_{01}. From Lemmas 5 and 7, it follows that $x_{01}(y_{01}z_{01})=0$ and so, $R_{01}R_{01}^2=0$. This is contrary to our assumption. The contradiction arises from the assumption that $R_{10}R_{01}=0$. Thus, $R_{10}R_{01}\neq 0$ and, hence, R_{11} must be associative.

We have proven

THEOREM. *Let R be a prime associator-dependent ring satisfying* (3) *where* $\alpha\neq -1, 1, -\frac{1}{2}, -2$ *and the characteristic of R is prime to* 6. *If R has an idempotent* $e\neq 0, 1$ *then R is alternative.*

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